Solving Knuth-Bendix Ordering Constraints

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We consider term algebras with simplification orders (which are monotonic, and well-founded). Solving constraints (quantifier-free formulas) in term algebras with this kind of orders has several important applications like pruning search space in automated deduction and proving termination of term rewriting systems. Two kinds of orders are normally used in automated deduction: the Knuth-Bendix order and various versions of recursive path orders. The Knuth-Bendix order is used in the state-of-the-art theorem provers, for example, E [12], SPASS [13], Vampire [11], and Waldmeister [3]. There is extensive literature on solving recursive path ordering constraints (e.g., [1,4,9,10]). The decidability of Knuth-Bendix ordering constraints was proved only recently in [6]. The algorithm described in that paper shows that the problem belongs to 2-NEXPTIME. It was also shown that the problem is NP-hard by reduction of the solvability of systems of linear Diophantine equations to the solvability of the Knuth-Bendix ordering constraints.

In our recent work we show that the existential theory of any term algebra with the Knuth-Bendix order is NP-complete [7]. Another question we consider is solvability of constraints consisting of a single inequality. For the lexicographic path orders this problem was shown to be NP-complete [2]. We show that solvability of the Knuth-Bendix ordering constraints consisting of a single inequality can be done in polynomial time. This algorithm can be consider as a part of a more complex algorithm for so called orientability problem: given a system of term rewrite rules decide whether there exists a Knuth-Bendix order which orients this system. We present a polynomial algorithm for the orientability problem for the Knuth-Bendix order [8].

Let us formulate our results more formally. We consider term algebras in a finite signature Σ with at least one constant, denoted $\mathrm{TA}(\Sigma)$. Let us now define Knuth-Bendix order on $\mathrm{TA}(\Sigma)$ [5]. The definition of Knuth-Bendix order is parametrized by a *weight function* on Σ, i.e., a function $w : \Sigma \to \mathbb{N}$, and a linear order $\gg$ on Σ. We require from the weight function the following: if $w(f) = 0$ and f is unary, then f must be the greatest w.r.t. $\gg$ in Σ, and weights of constants are positive. We define the weight of a ground term as a sum of weights of functors occurring in the term. Given a weight function w and a linear order $\gg$ on Σ, the *Knuth-Bendix order* on $\mathrm{TA}(\Sigma)$ is a binary relation $\succ$ defined as follows. For any ground terms $t = g(t_1, \dots, t_n)$ and $s = h(s_1, \dots, s_k)$ we have $t \succ s$ if one of the following conditions holds:

1. $w(t) > w(s)$;

2. $w(t) = w(s)$ and $g \gg h$;
3. $w(t) = w(s)$, $g = h$ and for some $1 \le i \le n$ we have $t_1 = s_1, \dots, t_{i-1} = s_{i-1}$ and $t_i \succ s_i$.

Theorem 1. *The existential first-order theory of any term algebra with the Knuth-Bendix order is NP-complete.*

Let us sketch some steps of our proof, for full details see [7]. To prove the theorem we first extend our term algebra with the natural numbers with addition and the weight function on terms. We show how to transform arbitrary quantifier-free formula into an equivalent disjunction of conjunctions such that all occurring terms in the formula are variables. Then we introduce formulas which express that there exists at least n terms of the weight x, where n is a fixed parameter and show how to write them as linear Diophantine equations on weights of terms. Using obtained formulas we transform initial formula into an equivalent constraint which consists of linear Diophantine equations on weights of terms. Then we show that satisfiability of that constraint is equivalent to satisfiability of the systems of linear Diophantine equations over the natural numbers.

Our next theorem is about solving constraints consisting of a single inequality.

Theorem 2. *There exists a polynomial time algorithm for solving Knuth-Bendix ordering constraints consisting of a single inequality.*

For a proof see [8]. Also in that paper was presented a polynomial time algorithm for the orientability problem.

Theorem 3. *The problem of the existence of a Knuth-Bendix order which orients a given rewrite rule systems can be solved in polynomial time.*

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