Unification and Projectivity in Propositional Logic

Silvio Ghilardi

Dipartimento di Scienze dell'Informazione
Università degli Studi,
Milano, Italy
ghilardi@dsi.unimi.it

Classical propositional calculus enjoys the following remarkable property (corresponding to the unitarity of E-unification type for Boolean algebras): for every formula A, if there is a substitution σ such that $\sigma(A)$ is provable, then there is also 'the best' substitution with this property, i.e. there is a substitution σ such that $\sigma(A)$ is provable and any τ such that $\tau(A)$ is provable is, up to provable equivalence, an instantiation of σ.

We wonder whether the same property holds for other logical calculi, for instance for some modal or (super)intuitionistic calculi. The answer is negative for all logics L enjoying disjunction property: for instance, in the modal case, the formula $\Box x \vee \Box \neg x$ has unifiers (by this we mean substitutions making it a theorem in L)

$$\sigma_1 : x \mapsto 1 \qquad \sigma_2 : x \mapsto 0$$

(where 1 is 'syntactic truth' and 0 is 'syntactic false') and there is no unifier more general than both of them because if

$$\vdash_L \Box \sigma(x) \vee \Box \neg\sigma(x)$$

then, by disjunction property, either $\vdash_L \sigma(x)$ (so that σ is equivalent to σ_1) or $\vdash_L \neg\sigma(x)$ (so that σ is equivalent to σ_2). However we prove that unification is rather nice in intuitionistic logic and in many common modal systems like $K4$, $S4$, $S4Grz$, GL, etc. because there are finitely many 'best unifiers' for any formula admitting at least a unifier. In more explicit terms, this means that the unification type of the corresponding classes of (Heyting, modal) algebras is finitary; notice in fact that for these classes of algebras, an arbitrary (elementary) E-unification problem

$$A_1 =_E A_1', \ \ldots, \ A_n =_E A_n'$$

is equivalent to the (matching) problem

$$(A_1 \leftrightarrow A_1') \wedge \cdots \wedge (A_n \leftrightarrow A_n') =_E 1$$

and hence to the problem of making a single formula a theorem in the corresponding logical calculus.

The area of E-unification in propositional logic is nevertheless new, many problems still wait for a solution or for a better solution: these problems include extensions to unification with constants and to general unification, investigation of further natural systems like K, structural information concerning the relationship between unification types and the the lattice of varieties, improvements of the few existing algorithms from the computational point of view, complexity questions, etc.

In my contribution I shall make a survey of my recent work in the field. In the first part, basic definitions are recalled, moreover an equivalent conceptual algebraic approach [1] (quite useful for the determination of unification types) is introduced and many rather easy examples are provided (almost all belonging to propositional logic). This algebraic approach relates unification theory with the standard notion of projective and of finitely presented object in a category. In the second part, the case of Heyting algebras is closely investigated [2]: formulas giving rise to finitely presented projective algebras are characterized both from a syntactic and a semantic point of view, moreover an algorithm [4] (based on a combination of tableaux and resolution) is introduced in order to recognize them. Suitably iterated, the algorithm determines a projective approximation of a given formula, thus providing a complete conformal unification algorithm. The basic point concerning intuitionistic unification is in fact the following: any unifier for a formula A turns out to be a unifier for a projective formula P implying A (and having at most the same implicational degree as A) and is consequently less general than the mgu of P (which trivially is also a unifier for A). Moreover, the comparison order between mgus of projective formulas coincides with IPC-provability among the projective formulas themselves. From this it follows that a complete set of unifiers of unifiers for A is completely determined by (any complete) set of maximal projective formulas implying A (i.e. by what we called a projective approximation of A).

Intuitionistic unification theory has important (and partially unexpected) logical applications to exact formulas and to admissible inference rules. Problems arising from first implementation experiences [5] are also discussed: here ah hoc weak (incomplete) theorem provers seem to be indispensable (in principle, they should be strong enough to reduce redundancies in the output sets but not so strong to cause large inefficiency).

In the final part of my contribution, I show the suitable adaptation of the strategy used in the intuitionistic case to the above mentioned modal systems [3]. Here the proof of the semantic and the syntactic characterization of projective formulas becomes more difficult: bounded bisimulations and ranks in K. Fine's style should be used from the beginning, moreover building mgus of projective formulas seems to require very large iterations of basic kinds of substitutions. However the remaining part of the plan goes as expected, thus yeldying finitarity results for unification types in this case too.

References

[1] S. Ghilardi *Unification through projectivity*, Journal of Logic and Computation, 7, 6, pp.733-752 (1997).
[2] S. Ghilardi *Unification in intuitionistic logic*, Journal of Symbolic Logic, vol.64, n.2, pp.859-880 (1999).
[3] S. Ghilardi *Best soving modal equations*, Annals of Pure and Applied Logic, 102, pp.183-198 (2000).
[4] S. Ghilardi *A resolution/tableaux algorithm for projective approximations in IPC*, Journal of Logic and Computation, to appear.
[5] B. Bertacco *Unificazione e Proiettività nella Logica Intuizionista Proposizionale: Realizzazione e Studio di un Algoritmo*, Tesi di Laurea in Informatica, Università degli Studi di Milano, a.a. 2000/2001.